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MODELING THE SOLID PHASE STRESS IN A FLUID–SOLID MIXTURE

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Abstract—An alternative formulation of the solid phase stress is given for a flowing fluid–solid mixture. This formulation allows us to treat all three sources of the solid phase stress with the same concept. This concept utilizes a control surface and considers stress as the force per unit area on such a surface. The three solid phase stress sources include the force between the particles, the rate of momentum transfer due to random particle motion and the hydrodynamic interaction between the particles and the fluid. Emphasis is put on the third of the three, because the first two are well-understood from the given concept. The hydrodynamic contribution is called the "particle-presence stress". The resulting formulation of this stress is identical to Batchelor's, in which a volume-averaging concept is used. The present result includes the particle inertia effect. It is also shown that the resulting solid phase pressure under special conditions reduces to a form proposed previously in the literature.

Key Words: two-phase, mixture, particle-presence, stress, fluid–solid, hydrodynamic, particle, micromechanics

1. INTRODUCTION

The mathematical model of a flowing fluid–solid mixture is far from complete. Current analyses of such flows commonly rely on semi-empirical formulas. The rheological parameters used in these formulas depend on the ranges of solid concentration and flow rate (Thomas 1965). It is unreliable to extrapolate these empirical formulas to situations beyond the test range. In order to more rigorously determine the rheological parameters, micromechanics has been utilized recently. Pioneer works using this approach include, among others, Batchelor (1970), Ishii (1975) and Drew (1983).

The micromechanics approach derives constitutive equations by analyzing the interactions of the fluid and solid constituent at the individual particle level. The result of this analysis is then averaged to obtain various macroscopic transport coefficients such as diffusivity and viscosity. In principle, if all the micromechanics at the individual particle level are incorporated, one should obtain transport coefficients explicitly and free from empirical constants. However, although theories have been successfully developed in this manner for a rapidly flowing solid dispersed in a vacuum, attempts to extrapolate this to a fluid–solid mixture have just begun.

When considering the fluid effect many difficulties are encountered. At low Reynolds number, the hydrodynamics in a mixture of high solid concentration is still an unsolved problem. Additional complications arise at high Reynolds number because of the existence of fluid turbulence and its interaction with freely moving solid particles. In view of the current understanding of these subjects, there is a long way to go before we can analytically derive the transport coefficients in a fluid–solid mixture flow. Nevertheless, research in this direction provides more insight and thus improves the existing semi-empirical formulas.

Two key steps in attempting a micromechanical approach are first, identifying and modeling the mechanisms to be incorporated; second, constructing an averaging method. The first identifies the flow regime to be studied and quantifies microscopic information, such as the forces between the interacting solid particles and the forces between a particle and the surrounding fluid. The second provides a mathematical framework to bridge the micro and macrobehavior of the fluid–solid mixture. Without a formal structure provided by the averaging method, confusion can easily arise when detailed mechanisms are to be incorporated. This confusion is seen when comparing several existing models for the stresses in a fluid–solid mixture flow (e.g. Ishii 1975; Drew 1983; McTigue *et al.* 1986).

The present work discusses the second step. Our ultimate goal is to provide a physically transparent and mathematically rigorous method. This method is used to model constitutive equations in the balance laws for a fluid–solid mixture.

The balance laws for the mass and momentum in flows of a fluid–solid mixture have been given by Ishii (1975) as follows:

$$\frac{\partial \rho^s}{\partial t} + \nabla \cdot (\rho^s \mathbf{u}) = 0 \qquad [1a]$$

$$\rho^s \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \rho^s \mathbf{g} + \mathbf{m} + \nabla \cdot (c \mathbf{T}^s) \qquad [1b]$$

and

$$\frac{\partial \rho^f}{\partial t} + \nabla \cdot (\rho^f \mathbf{v}) = 0 \qquad [2a]$$

$$\rho^f \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho^f \mathbf{g} - \mathbf{m} + \nabla \cdot [(1 - c) \mathbf{T}^f]. \qquad [2b]$$

In the above, $\rho^s = \rho_s c$ and $\rho^f = \rho_f(1 - c)$ are the partial densities of the solid and fluid phase, respectively, where c is the solid concentration; ρ_s and ρ_f are the solid and fluid material density, respectively; the mass-weighted average of the solid and fluid velocity are $\mathbf{u}$ and $\mathbf{v}$, respectively; the phase interaction force per unit volume is denoted by $\mathbf{m}$ and the mass-weighted stress for the solid and fluid phase are $\mathbf{T}^s$ and $\mathbf{T}^f$, respectively.

In this paper, we discuss the modeling of $\mathbf{T}^s$ for an interacting fluid–solid mixture. A similar procedure may be applied to model other constitutive equations in the solid momentum and energy equation. This will be the subject of a future study.

2. MODELING THE SOLID PHASE STRESS

Consider an arbitrary control volume V in a fluid–solid mixture, as shown in figure 1. This mixture consists of a Newtonian fluid and uniform spherical particles. This control volume may be decomposed into V_s and V_f representing the solid and fluid portion, respectively. The solid portion V_s is shaded in figure 1. The corresponding control surface S may also be decomposed into two portions S_s and S_f representing the parts of S occupied by the solid and fluid phase, respectively. The solid portion S_s is shown as thick black lines in figure 1.

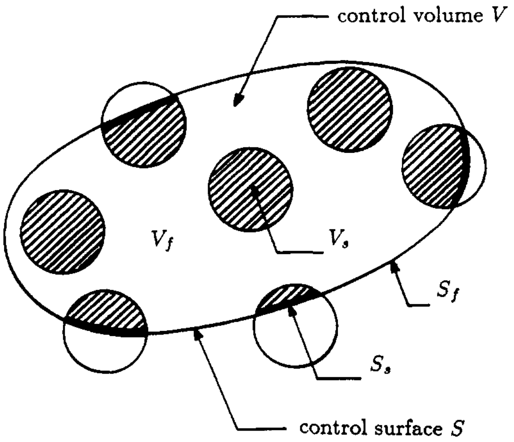


Figure 1. A control volume in the mixture.

Total surface force acting on S and total body force acting on V are responsible for the change of linear momentum inside V. The two-phase approach adopted by Ishii (1975) and many others requires separation of the total stress into the solid and fluid phase stress $\mathbf{T}^s$ and $\mathbf{T}^f$ introduced in [1b] and [2b]. If $\hat{\mathbf{N}}$ is the local unit normal of S, $\mathbf{T}^s \cdot \hat{\mathbf{N}}$ is the force acting on S_s per unit area of S_s and $\mathbf{T}^f \cdot \hat{\mathbf{N}}$ is the force acting of S_f per unit area of S_f.

From fluid mechanics,

$$\mathbf{T}^f = \mathbf{T}^v + \mathbf{T}^t \qquad [3]$$

where $\mathbf{T}^v$ and $\mathbf{T}^t$ are the viscous and turbulent stress acting on S_f from the fluid motion. The solid phase stress $\mathbf{T}^s$, in general, may come from the momentum transfer of solid particle collisions, the random motion of the solid particles and the hydrodynamic force acting on the surface of the particles. Thus,

$$\mathbf{T}^s = \mathbf{T}^c + \mathbf{T}^k + \mathbf{T}^p. \qquad [4]$$

The physical meanings of these three components are given below.

The collisional stress $\mathbf{T}^c$ is the rate of momentum transfer across S_s per unit area of S^s due to particle collisions. Bagnold (1954) modeled this stress as

$$\mathbf{T}^c = f\mathscr{P}\Delta M, \qquad [5]$$

where f is the particle collision frequency, $\mathscr{P}$ is the average number of particles cut by unit area of S and ΔM is the average momentum transfer per collision. Later works (e.g. Jenkins & Savage 1983; Lun *et al.* 1984) further modified the formulation of $\mathbf{T}^c$ based on the kinetic theory of gases, but the basic concept remains unchanged. The kinetic stress $\mathbf{T}^k$ is the momentum transfer due to the random motion of solid particles and has been modeled as (e.g. Lun *et al.* 1984; Babič 1985)

$$\mathbf{T}^k = \langle \rho_s \mathbf{u}''\mathbf{u}'' \rangle, \qquad [6]$$

where $\langle\ \rangle$ is the ensemble average, $\mathbf{u}''$ is the random velocity of particles defined by the difference of a single velocity realization of a particle and the mass-weighted average velocity of the solid phase. Here, we have tacitly assumed that the balance equations [1a, b] and [2a, b] are time-averaged with respect to a very small time scale over which macroscopic properties are stationary. Without doing so, transfer of momentum during instantaneous collisions can not be captured. A similar time scale has been used to derive the Reynolds stress from turbulence $\mathbf{T}^t$.

The hydrodynamic contribution of the solid phase stress, $\mathbf{T}^p$, is less understood from the above viewpoint. That is, a formulation of $\mathbf{T}^p$ as the force per unit area on S_s due to the fluid force surrounding it is not available in the literature. Batchelor (1970) gave a mathematical derivation of $\mathbf{T}^p$, defining it as the volume average of stress residing in the solid particles. For completeness, this derivation is repeated in appendix A. Batchelor's approach provides a very short and elegant formulation. It easily handles the hydrodynamic contribution of the solid phase stress. But conceptually it is hard to extend this approach to include the other two components $\mathbf{T}^c$ and $\mathbf{T}^k$. This is because these other two stress components are strickly modeled from the momentum transfer rate across a control surface, which is a different concept from the volume-averaging of internal solid stress. The alternative formulation of $\mathbf{T}^p$ given here is based on a control surface concept identical to what is used to obtain $\mathbf{T}^c$ and $\mathbf{T}^k$.

We will call $\mathbf{T}^p$ the "particle-presence" stress, because this stress is not a result of the particles' motion, but is rather a result of their existence in the fluid flow. Their rigid boundary alters the local fluid flow pattern and thus changes the bulk rheological property.

The existence of $\mathbf{T}^p$ is easily seen if we consider a dilute slow simple shear flow with vanishing particle Reynolds number. This flow has been studied by Einstein (1906) and Batchelor & Green (1972) to give a total shear stress

$$T_{ij} = (1-c)T^f_{ij} + cT^s_{ij} = 2\mu(1+2.5c)E_{ij}, \qquad [7]$$

where μ is the fluid viscosity and E_{ij} is the bulk strain-rate, defined in appendix B. We now study the five components in [3] and [4] for this flow. In this flow, fluid is laminar, hence $\mathbf{T}^t = 0$. The

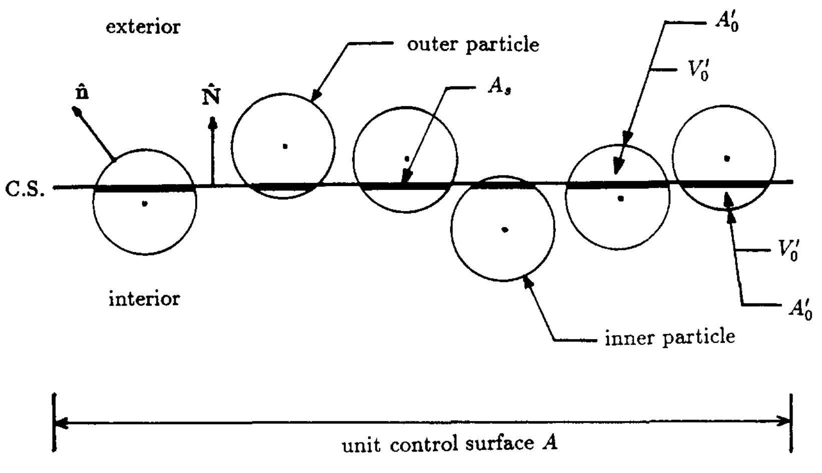


Figure 2. Definition sketch of a unit control surface.

concentration approaches zero so that there is no particle collision, consequently $\mathbf{T}^c = 0$. Moreover, in this flow, particles are driven by a laminar fluid motion, hence they do not fluctuate if the body force is stationary. In the absence of particle collision induced random motion, this implies $\mathbf{T}^k = 0$. At this point, only $\mathbf{T}^v$ and $\mathbf{T}^p$ are still present. For the Newtonian fluid considered here, the shear stress $T^v_{ij} = 2\mu\{e^f_{ij}\}$, where e^f_{ij} is the local fluid shear strain rate, $\{\ \}$ indicates the mass-weighted average. In appendix B, it is shown that $(1-c)\{e^f_{ij}\} = E_{ij}$. Therefore $(1-c)T^f_{ij} = 2\mu E_{ij}$. The rest of the terms in [7], i.e. $5c\mu E_{ij}$, must come from the only remaining solid phase stress $\mathbf{T}^p$.

Let us consider a unit surface area A on S with unit normal $\hat{\mathbf{N}}$, as shown in figure 2. Denote the portion of this unit area occupied by particles as A_s, the thick black lines in figure 2. A total of $\mathscr{P}$ particles lie on A. These are called "surface particles". All surface particles with their center inside the control volume are called "inner particles", and surface particles with their center exterior to the control volume are called "outer particles". It should be noted that an outer particle still has some fraction of its volume inside the control volume. Similarly, an inner particle has some fraction of its volume exterior to the control volume. If particles are randomly distributed in such a way that the number density of particles does not vary in a length scale equal to or greater than the particle's largest dimension, then, on average, there are an equal number, i.e. $\mathscr{P}/2$, of inner and outer particles on a unit control area.

The hydrodynamic contribution of the solid phase stress is the force acting on A_s due to the fluid force on the surface of the surface particles. Since A_s lies inside the particles, it would seem that one needs to know the stress distribution inside these particles to obtain $\mathbf{T}^p$. This is in fact unnecessary. The alternative is to relate the force acting inside the particle to the force acting on the surface of the particle. The total surface force **sf**, which includes both the normal and tangential component, acting on the solid portion of the unit control surface can be expressed as follows:

$$\mathbf{sf} = \sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}^i_p - \sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}^o_p - \sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}^i_t + \sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}^o_t, \qquad [8]$$

where, as shown in figure 3, $\mathbf{s}^i_p$ and $\mathbf{s}^o_p$ are the partial surface forces acting on the curved area of the inner and outer particles, respectively; and $\mathbf{s}^i_t$ and $\mathbf{s}^o_t$ are the total surface force acting on the particle fraction of the inner and outer particles, respectively. For graphical simplicity, only the tangential components of all forces are shown in figure 3.

A unit control surface cuts through, on average, an equal number, $\mathscr{P}/2$, of inner and outer particles. Consider the inner particles for now, each of them has a different amount of its surface area exposed to the exterior fluid, as shown in figure 2. For any given $\hat{\mathbf{n}}$, shown in figure 4, such that $\hat{\mathbf{n}} \cdot \hat{\mathbf{N}} \geqslant 0$, there is an infinitesimal surface area on an inner particle associated with this direction. This area may or may not be exposed to the exterior fluid. If it is, then the hydrodynamic

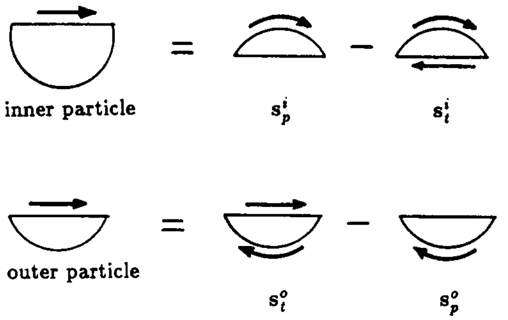


Figure 3. Various types of surface forces.

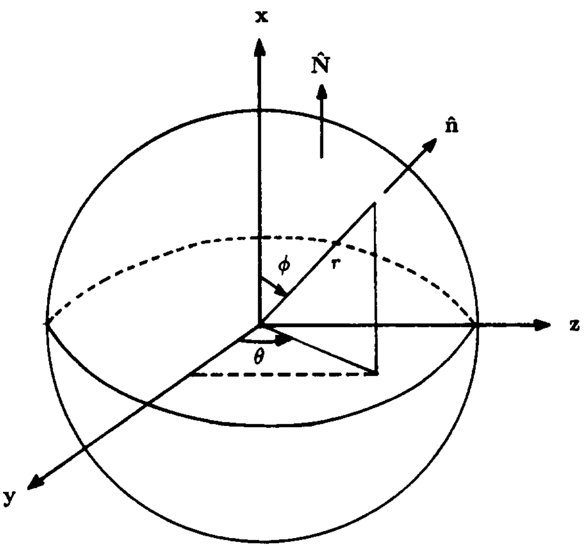


Figure 4. Coordinate systems.

force acting on it contributes to $\mathbf{s}_\mathrm{p}^\mathrm{i}$ and $\mathbf{s}_\mathrm{t}^\mathrm{i}$. Assuming an isotropic distribution of particles, the depth with which a random control surface S cut through a particle is uniformly distributed along its diameter. Therefore, the probable number of inner particles on a unit control surface with the infinitesimal area associated with $\hat{\mathbf{n}}$ exposed to the exterior fluid is $(\mathscr{P}/2)\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}$. From figure 4,

$$\frac{\mathscr{P}}{2}\hat{\mathbf{n}}\cdot\hat{\mathbf{N}} = \frac{\mathscr{P}}{2}\cos\phi. \qquad [9]$$

Similarly, for a given $\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}\cdot\hat{\mathbf{N}} \leqslant 0$, the probable number of outer particles on a unit area with the infinitesimal area associated with $\hat{\mathbf{n}}$ exposed to the interior fluid is

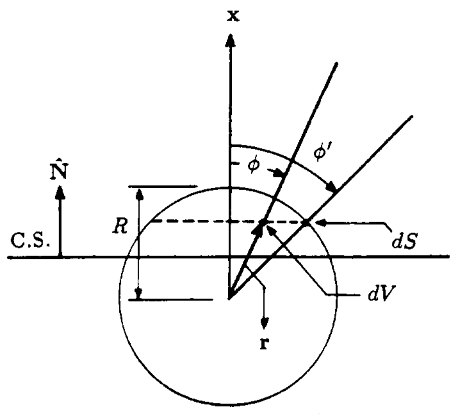
$$-\frac{\mathscr{P}}{2}\hat{\mathbf{n}}\cdot\hat{\mathbf{N}} = -\frac{\mathscr{P}}{2}\cos\phi. \qquad [10]$$

Utilizing [9] and [10] and recognizing that the infinitesimal surface area of a sphere with radius R is $R^2 \sin\phi \,\mathrm{d}\theta\,\mathrm{d}\phi$, one may obtain

$$\begin{aligned}\sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}_\mathrm{p}^\mathrm{i} - \sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}_\mathrm{p}^\mathrm{o} &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} (\Sigma_{ik}n_k)\left(\frac{\mathscr{P}}{2}\cos\phi\right) R^2\,\mathrm{d}\theta \sin\phi\,\mathrm{d}\phi \\ &\quad - \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} (\Sigma_{ik}n_k)\left(-\frac{\mathscr{P}}{2}\cos\phi\right) R^2\,\mathrm{d}\theta \sin\phi\,\mathrm{d}\phi && [11] \\ &= \int_{\frac{A_0}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}\geqslant 0)} \Sigma_{ik}n_\mathrm{k}\left(\frac{\mathscr{P}}{2}n_jN_j\right)\mathrm{d}A - \int_{\frac{A_0}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}\leqslant 0)} \Sigma_{ik}n_k\left(-\frac{\mathscr{P}}{2}n_jN_j\right)\mathrm{d}A, && [12]\end{aligned}$$

where Σ_{ik} is the fluid stress component on the surface of a particle and A_0 is the surface area of a particle. Let n be the solid particles' number density per unit volume of the mixture, it has been shown that $\mathscr{P} = 2nR$ (Shen & Ackerman 1982). Using this, [12] becomes

$$\begin{aligned}\sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}_\mathrm{p}^\mathrm{i} - \sum_1^{\frac{\mathscr{P}}{2}} \mathbf{s}_\mathrm{p}^\mathrm{o} &= \left(n\int_{\frac{A_0}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}\geqslant 0)} \Sigma_{ik}n_kr_j\,\mathrm{d}A\right)N_j - \left(-n\int_{\frac{A_0}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}\leqslant 0)} \Sigma_{ik}n_kr_j\,\mathrm{d}A\right)N_j && [13] \\ &= \left(n\int_{A_0} \Sigma_{ik}n_kr_j\,\mathrm{d}A\right)N_j. && [14]\end{aligned}$$



$$r\cos\phi = R\cos\phi'$$

$$\cos\phi' = \frac{r}{R}\cos\phi$$

Figure 5. Corresponding relation between dV and dS.

The total surface surface force, $\mathbf{s}_t^i$ and $\mathbf{s}_t^o$, acting on the particle fractions can be obtained by relating them to $\nabla\cdot\Sigma$ inside the particle using Gauss' theorem. That is,

$$\int_{A_0'} \Sigma_{ik} n_k \, \mathrm{d}A = \int_{V_0'} \partial_k \Sigma_{ik} \, \mathrm{d}V \qquad [15]$$

where A_0' and V_0', as shown in figure 2, are the total area and volume of the particle fraction outside the control volume for an inner particle or inside the control volume for an outer particle. Consider first an inner particle, as shown in figure 5. Let **r** be a position vector originating from the center of the particle. An infinitesimal volume dV locates at the tip of **r**. The probability of this dV being exterior to the control volume is identical to the probability of having a surface area "at the same level" as dV exposed to the exterior fluid. As in the development of [9], one finds that the probable number of inner particles on a unit control surface having a given dV exterior to the control volume is

$$\frac{\mathscr{P}}{2}\cos\phi' = \frac{\mathscr{P}}{2}\frac{r}{R}\cos\phi = nr\cos\phi, \qquad [16]$$

where the angles ϕ and ϕ' are defined in figure 5 and r is the distance from center of the particle to the infinitesimal volume dV. In a similar way, the probable number of outer particles in a unit area which have a given dV inside the control volume is $-nr\cos\phi$. Utilizing the above observation and [15] and [16],

$$-\sum_{1}^{\frac{\mathscr{P}}{2}} \mathbf{s}_t^i + \sum_{1}^{\frac{\mathscr{P}}{2}} \mathbf{s}_t^o = -\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^R \partial_k \Sigma_{ik}(nr\cos\phi) r^2 \, \mathrm{d}r \, \mathrm{d}\theta \sin\phi \, \mathrm{d}\phi$$

$$+\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\int_0^R \partial_k \Sigma_{ik}(-nr\cos\phi) r^2 \, \mathrm{d}r \, \mathrm{d}\theta \sin\phi \, \mathrm{d}\phi \qquad [17]$$

$$= -\int_{\frac{V_0}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}\geqslant 0)} \partial_k \Sigma_{ik}(nr_j N_j) \, \mathrm{d}V + \int_{\frac{V_0}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}\leqslant 0)} \partial_k \Sigma_{ik}(-nr_j N_j) \, \mathrm{d}V \qquad [18]$$

$$= -n\left(\int_{V_0} \partial_k \Sigma_{ik} r_j \, \mathrm{d}V\right) N_j, \qquad [19]$$

where V_0 is the volume of a particle. From [8], [14] and [19], the total surface force on the intersectional area A_s per unit control surface can thus be formulated as

$$sf_i = n\left(\int_{A_0} \Sigma_{ik} n_k r_j \, \mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} r_j \, \mathrm{d}V\right) N_j. \qquad [20]$$

Therefore, the final formulation of the particle-presence stress is

$$T^{\mathrm{p}}_{ij}=\frac{n}{c}\left(\int_{A_0}\Sigma_{ik}n_k r_j\,\mathrm{d}A-\int_{V_0}\partial_k\Sigma_{ik}r_j\,\mathrm{d}V\right) \qquad [21]$$

$$=\frac{1}{V_0}\left(\int_{A_0}\Sigma_{ik}n_k r_j\,\mathrm{d}A-\int_{V_0}\partial_k\Sigma_{ik}r_j\,\mathrm{d}V\right), \qquad [22]$$

which is identical to Batchelor's (1970) result for a slow flow of a dilute fluid–solid mixture. However, this derivation clearly reveals the physical origin of the particle-presence stress as well as providing a consistent procedure to incorporate the other two solid phase stresses $\mathbf{T}^{\mathrm{c}}$ and $\mathbf{T}^{\mathrm{k}}$.

The above analysis may be extended straightforwardly to arbitrary shape particles. The formulation given by [22] remains the same if the orientation of the particles is uniformly distributed. Otherwise, the probability of the orientation distribution will enter the stress-averaging in [22].

3. COROLLARIES OF THE PARTICLE-PRESENCE STRESS

We now take the general formulation of the particle-presence stress given in [22] and obtain two specific results. These two results, to our knowledge, have not been rigorously obtained previously.

First, let us obtain the expression for the second term on the r.h.s. of [22]. This term is associated with the particle's inertia and was neglected by Batchelor & Green (1972). In that work, they studied a slow simple shear flow of a dilute fluid and spherical particle mixture. By integrating the stress distribution of a Stokes flow around a sphere, they showed that

$$\frac{1}{V_0}\int_{A_0}\Sigma_{ik}n_k r_j\,\mathrm{d}A=5\mu E_{ij}, \qquad [23]$$

which gave Einstein's result. In the case of a slow dilute flow, any particle inertia is indeed negligible. In general this may not be true. Applying the equation of motion to an infinitesimal volume inside the solid phase shows that

$$\partial_k\Sigma_{ik}=\rho_{\mathrm{s}}\dot{V}^{\mathrm{l}}_i-\rho_{\mathrm{s}}g_i \qquad [24]$$

where the overdot represents the usual time derivative for a rigid body, V^{l}_i is the local velocity of a particular point inside the solid particle and g_i is the body force per unit mass. Substituting [24] into the second term on the r.h.s. of [22] yields

$$\int_{V_0}\partial_k\Sigma_{ik}r_j\,\mathrm{d}V=\rho_{\mathrm{s}}\int_{V_0}(\dot{V}^{\mathrm{l}}_i-g_i)r_j\,\mathrm{d}V$$

$$=\left(\rho_{\mathrm{s}}\int_{V_0}\dot{V}^{\mathrm{l}}_i r_j\,\mathrm{d}V-\rho_{\mathrm{s}}g_i\int_{V_0}r_j\,\mathrm{d}V\right)$$

$$=\rho_{\mathrm{s}}\int_{V_0}\dot{V}^{\mathrm{l}}_i r_j\,\mathrm{d}V. \qquad [25]$$

The local velocity inside a rigid particle is

$$\mathbf{V}^{\mathrm{l}}=\mathbf{V}+\boldsymbol{\Omega}+\mathbf{r}, \qquad [26]$$

where $\mathbf{V}$ is the velocity of the center of the particle and $\boldsymbol{\Omega}$ is the angular velocity of the particle. The derivative of [26] yields

$$\dot{\mathbf{V}}^{\mathrm{l}}=\dot{\mathbf{V}}+\dot{\boldsymbol{\Omega}}\times\mathbf{r}+\boldsymbol{\Omega}\times\dot{\mathbf{r}}$$

$$=\dot{\mathbf{V}}+\dot{\boldsymbol{\Omega}}\times\mathbf{r}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}). \qquad [27]$$

Changing the above into the indicial notation and simplifying, [27] becomes

$$\dot{V}^{\mathrm{l}}_i = \dot{V}_i + \epsilon_{ilk}\dot{\Omega}_l r_k + \Omega_k\Omega_i r_k - \Omega_k\Omega_k r_i. \qquad [28]$$

Substituting [28] into [25] yields

$$\begin{aligned}\int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V &= \rho_{\mathrm{s}} \int_{V_0} (\dot{V}_i r_j + \epsilon_{ilk}\dot{\Omega}_l r_k r_j + \Omega_k\Omega_i r_k r_j - \Omega_k\Omega_k r_i r_j)\,\mathrm{d}V \\ &= \rho_{\mathrm{s}}\left(\dot{V}_i \int_{V_0} r_j \,\mathrm{d}V + \epsilon_{ilk}\dot{\Omega}_l \int_{V_0} r_k r_j \,\mathrm{d}V + \Omega_k\Omega_i \int_{V_0} r_k r_j \,\mathrm{d}V - \Omega_k\Omega_k \int_{V_0} r_i r_j \,\mathrm{d}V\right). \qquad [29]\end{aligned}$$

Since

$$\int_{V_0} r_i r_j \,\mathrm{d}V = \tfrac{4}{15}\pi R^5 \delta_{ij}, \qquad [30]$$

[29] may be simplified to yield

$$\begin{aligned}\int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V &= \frac{4\pi}{15}\rho_{\mathrm{s}} R^5 (\epsilon_{ilk}\dot{\Omega}_l \delta_{kj} + \Omega_k\Omega_i\delta_{kj} - \Omega_k\Omega_k\delta_{ij}) \\ &= \frac{4\pi}{15}\rho_{\mathrm{s}} R^5 (\epsilon_{ilj}\dot{\Omega}_l + \Omega_i\Omega_j - \Omega_k\Omega_k\delta_{ij}). \qquad [31]\end{aligned}$$

Hence the second term in [22] becomes

$$\frac{1}{V_0}\int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V = \tfrac{1}{5}\rho_{\mathrm{s}} R^2 (\epsilon_{ilj}\dot{\Omega}_l + \Omega_i\Omega_j - \Omega_k\Omega_k\delta_{ij}). \qquad [32]$$

The above is the contribution of particle inertia to T^{p}_{ij}. For a free spherical particle in an infinite fluid, $\Omega_i = \frac{1}{2}\epsilon_{ijk}\partial_j v_k$, where v_k is the fluid velocity. Hence, by comparing [23] and [32], for a dilute system in the Stokes regime such that $(\rho_{\mathrm{f}} R^2 \partial_j v_k)/\mu \ll 1$, the second term in [22] is negligible when compared with the first term.

Secondly, the solid phase pressure from $\mathbf{T}^{\mathrm{p}}$ will be obtained. This pressure has been proposed by Givler (1987) as the average fluid pressure around a particle's surface. Using the definition in continuum mechanics, the dynamic pressure p^{c} of any continuum is

$$p^{\mathrm{c}} = -\tfrac{1}{3}T^{\mathrm{c}}_{ii} \qquad [33]$$

where T^{c}_{ij} is the stress tensor in the continuum. If one ignores the second term in [22], the above yields

$$p^{\mathrm{p}} = -\frac{R^3}{3V_0}\int_0^{2\pi}\int_0^{\pi} \Sigma_{ik} n_k n_i \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta, \qquad [34]$$

where p^{p} is the pressure force from T^{p}. Substituting the Newtonian fluid stress into [34] yields

$$p^{\mathrm{p}} = -\frac{R^3}{3V_0}\int_0^{2\pi}\int_0^{\pi} (-p\delta_{ik} - \tfrac{2}{3}\mu e_{ll}\delta_{ik} + 2\mu e_{ik}) n_k n_i \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta \qquad [35]$$

where p is the fluid pressure. Due to the incompressibility of the fluid the second term in the parentheses in [35] is zero. For a slow flow of a dilute fluid–solid mixture, the third term of the integrand vanishes, as shown in appendix C. For this type of flow, the solid phase pressure reduces to

$$p^{\mathrm{p}} = \frac{R^3}{3V_0}\int_0^{2\pi}\int_0^{\pi} p \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta, \qquad [36]$$

which is the average hydrodynamic pressure on the surface of a particle, identical to what Givler (1987) has proposed. However, in a general case, this result may be modified when the third term of the integrand in [35] does not vanish.

4. CONCLUSION

A physical explanation of the particle-presence stress is given first. This stress is from the hydrodynamic interaction between the fluid and solid phases in a flowing mixture. A mathematical derivation of the stress is then provided in a consistent manner with which the other solid phase stresses are derived. This derivation uses the concept that stresses are forces per unit area on the control surface of an arbitrary control volume. The final result is shown to be identical to Batchelor's (1970)—in which a volume-averaging method was used and the formulation was conceptually different from what is given here. Lastly, the final result of the particle-presence stress is studied to include an expression for the inertia contribution, which may become important in the case of a nonvanishing particle Reynolds number. This inertia formulation has not been reported previously in the literature. The solid phase pressure from the hydrodynamic force contribution is also derived from the definition in continuum mechanics. The result is identical to a previously proposed form for a slow flow of a dilute fluid and spherical particle mixture. In general, additional terms that have not been discovered previously may also be present.

The approach given in this work is, we believe, physically transparent and also mathematically rigorous. It can be easily extended to model other constitutive relations in a two-phase flow. This will be the subject of future studies.

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APPENDIX A

This appendix gives the derivation by Batchelor (1970) of the solid phase stress in a dilute fluid–solid mixture where particle's collisions are ignored.

Let x_j be components of the position vector of an infinitesimal volume $\mathrm{d}V$ inside a particle of volume V_0, and Σ_{ik} be the components of the stress tensor in the particle. The following may be derived:

$$\int_{V_0} \partial_k \Sigma_{ik} x_j \,\mathrm{d}V = \int_{V_0} \partial_k(\Sigma_{ik} x_j)\,\mathrm{d}V - \int_{V_0} \Sigma_{ik}\partial_k x_j \,\mathrm{d}V \qquad [A.1]$$

$$= \int_{A_0} \Sigma_{ik} x_j n_k \,\mathrm{d}A - \int_{V_0} \Sigma_{ik}\delta_{kj}\,\mathrm{d}V \qquad [A.2]$$

$$= \int_{A_0} \Sigma_{ik} n_k x_j \,\mathrm{d}A - \int_{V_0} \Sigma_{ij}\,\mathrm{d}V \qquad [A.3]$$

or

$$\int_{V_0} \Sigma_{ij}\,\mathrm{d}V = \int_{A_0} \Sigma_{ik} n_k x_j \,\mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} x_j \,\mathrm{d}V, \qquad [A.4]$$

where A_0 is the surface of V_0. Let V_r be a representative volume inside a fluid–solid mixture. The linear dimension of V_r is determined in such a way that it should be sufficiently large compared with the particle spacing. On the other hand, it should be small enough to provide a uniform flow field inside V_r, within which there are a number of identical particles each of volume V_0. The solid phase stress, defined as the average stress in all the solid particles per unit volume of solid, is therefore,

$$T^{\mathrm{p}}_{ij} = \frac{1}{\sum V_0}\sum \int_{V_0} \Sigma_{ij}\,\mathrm{d}V = \frac{1}{\sum V_0}\sum\left(\int_{A_0} \Sigma_{ik} n_k x_j \,\mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} x_j \,\mathrm{d}V\right),$$

where the Σ is over all particles in V_r. The above may be further reduced as follows. Let $x_j = P_j + r_j$, where P_j are the components of the position vector of the center of a particle. Substituting this into the above gives

$$\begin{aligned} T^{\mathrm{p}}_{ij} &= \frac{1}{\sum V_0}\sum\left(\int_{A_0} \Sigma_{ik} n_k x_j \,\mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} x_j \,\mathrm{d}V\right) \\ &= \frac{1}{\sum V_0}\sum\left(\int_{A_0} \Sigma_{ik} n_k r_j \,\mathrm{d}A + \int_{A_0} \Sigma_{ik} n_k P_j \,\mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V - \int_{V_0} \partial_k \Sigma_{ik} P_j \,\mathrm{d}V\right) \\ &= \frac{1}{\sum V_0}\sum\left(\int_{A_0} \Sigma_{ik} n_k r_j \,\mathrm{d}A + \int_{V_0} \partial_k(\Sigma_{ik} P_j)\,\mathrm{d}V - \int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V - \int_{V_0} \partial_k \Sigma_{ik} P_j \,\mathrm{d}V\right) \\ &= \frac{1}{\sum V_0}\sum\left(\int_{A_0} \Sigma_{ik} n_k r_j \,\mathrm{d}A + \int_{V_0} \partial_k \Sigma_{ik} P_j \,\mathrm{d}V + \int_{V_0} \Sigma_{ik} \partial_k P_j \,\mathrm{d}V \right. \qquad [A.5] \\ &\quad \left. - \int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V - \int_{V_0} \partial_k \Sigma_{ik} P_j \,\mathrm{d}V\right) \\ &= \frac{1}{\sum V_0}\sum\left(\int_{A_0} \Sigma_{ik} n_k r_j \,\mathrm{d}A - \int_{V_0} \partial_k \Sigma_{ik} r_j \,\mathrm{d}V\right). \qquad [A.6] \end{aligned}$$

The third term in [A.5] is identical to zero, because $\partial_k P_j = 0$ for any one particle. Since the uniformity condition is assumed in this representative volume V_r, $(\Sigma V_0)/V = c$ and the summation is over a total of nV identities. Therefore [A.6] is identical to [21].

APPENDIX B

This appendix derives the bulk strain-rate tensor in a fluid–solid mixture.

By definition, the bulk-strain components related to the fluid and solid strain-rate are as follows:

$$E_{ij} = \langle e_{ij} \rangle \qquad [B.1]$$

$$= \langle (1-c) e^{\mathrm{f}}_{ij} \rangle + \langle c e^{\mathrm{s}}_{ij} \rangle \qquad [B.2]$$

$$= \langle 1-c \rangle \{ e^{\mathrm{f}}_{ij} \} + \langle c \rangle \{ e^{\mathrm{s}}_{ij} \}. \qquad [B.3]$$

In the above, c is the solid concentration for a single realization, $c = 0$ if fluid is sampled and $c = 1$ if solid is sampled, the ensemble $\langle c \rangle$ is the bulk solid concentration and is denoted by c in the text; e_{ij} is the local strain-rate from a single realization; e^{f}_{ij} and e^{s}_{ij} are the local fluid and solid strain-rate, respectively. The above may also be written in terms of the local gradients of the fluid and solid velocity $\mathbf{v}$ and $\mathbf{u}$, respectively, as

$$E_{ij} = \frac{\langle 1-c \rangle}{2} \left\{ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\} + \frac{\langle c \rangle}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\}. \qquad [B.4]$$

For a rigid body, $e^{\mathrm{s}}_{ij} = 0$, therefore

$$E_{ij} = \langle 1-c \rangle \{ e^{\mathrm{f}}_{ij} \}. \qquad [B.5]$$

This definition of the bulk strain-rate was used by Batchelor (1970).

It is important to realize that there is at least one other way to define the bulk strain-rate **E**. Namely, one would first define the bulk flow velocity as

$$U_i = \langle (1-c) v_i \rangle + \langle c u_i \rangle \qquad [B.6]$$

$$= \langle (1-c) \rangle \{ v_i \} + \langle c \rangle \{ u_i \}, \qquad [B.7]$$

where v_i and u_i are the components of local fluid and solid velocity, respectively. The bulk strain-rate components may thus be defined as

$$E^{\mathrm{a}}_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \qquad [B.8]$$

where the superscript "a" denotes "alternative". To simplify the matter, let us assume a uniform solid concentration. Equation [B.8] becomes

$$E^{\mathrm{a}}_{ij} = \frac{\langle 1-c \rangle}{2} \left(\frac{\partial \{v_i\}}{\partial x_j} + \frac{\partial \{v_j\}}{\partial x_i} \right) + \frac{\langle c \rangle}{2} \left(\frac{\partial \{u_i\}}{\partial x_j} + \frac{\partial \{u_j\}}{\partial x_i} \right). \qquad [B.9]$$

In general, the two definitions given in [B.4] and [B.9] are not identical. Because although the solid particles do not deform, the mass-weighted average velocity of the solid phase may have nonvanishing gradients.

In the literature, both definitions have been used. It is extremely important to clarify which is meant in a given paper before using the results. In this paper, [B.3] (or equivalently [B.4]) is the definition for the bulk strain-rate in a mixture flow. For the purpose of computing bulk stress, the mass weighted average fluid and solid stress must first be obtained. The mass-weighted average fluid stress is obtained from the local fluid stress which is a function of the local fluid strain-rate e_{ij}, not the bulk strain-rate E_{ij}. In view of this, when calculating the stress, definition of the bulk strain-rate given by [B.3] or [B.4] is physically more relevant.

APPENDIX C

This appendix shows that the third term on the r.h.s. of the integrand of [35] is zero for a Stokes flow.

Based on Batchelor's expression for the local strain-rate tensor, one can prove that $e_{ik}n_kn_i$ is actually identical to zero at $r = R$:

$$e_{ik}n_kn_i = E_{ik}n_kn_i\left(1-\frac{R^5}{r^5}\right) + E_{ml}\left(\frac{r_ir_m\delta_{kl}n_kn_i + r_kr_m\delta_{il}n_kn_i}{r^2} - \frac{2}{3}\frac{r_mr_l}{r^2}\delta_{ik}n_kn_i\right)\left(-\frac{5R^3}{2r^3}+\frac{5R^5}{r^5}\right)$$

$$+ E_{ml}\frac{r_mr_l}{r^2}\left(\frac{r_ir_kn_kn_i}{r^2} - \tfrac{1}{3}\delta_{ik}n_kn_i\right)\left(\frac{25R^5}{2r^5}\right)$$

$$= E_{ik}n_kn_i\left(1-\frac{R^5}{r^5}\right) + E_{ml}\left(\frac{rr_mn_l + rr_mn_l}{r^2} - \frac{2}{3}\frac{r_mr_l}{r^2}\delta_{ik}n_kn_i\right)\left(-\frac{5R^3}{2r^3}+\frac{5R^5}{r^5}\right)$$

$$+ \tfrac{2}{3}E_{ml}\frac{r_mr_l}{r^2}\left(\frac{25R^3}{2r^3} - \frac{35R^5}{2r^5}\right)$$

$$= E_{ik}n_kn_i\left[1-\frac{R^5}{r^5}+\frac{4}{3}\left(-\frac{5R^3}{2r^3}+\frac{5R^5}{r^5}\right)+\frac{2}{3}\left(\frac{25R^3}{2r^3}-\frac{35R^5}{2r^5}\right)\right].$$

The last expression is equal to zero when $r = R$.